

7.2 - Inverse Transforms and Transforms of Derivatives

Theorem 7.2.1: Some Inverse Transforms

$$(a) 1 = \mathcal{L}^{-1} \left\{ \frac{1}{s} \right\}$$

$$(b) t^n = \mathcal{L}^{-1} \left\{ \frac{n!}{s^{n+1}} \right\}, n = 1, 2, 3, \dots$$

$$(c) e^{at} = \mathcal{L}^{-1} \left\{ \frac{1}{s-a} \right\}$$

$$(d) \sin kt = \mathcal{L}^{-1} \left\{ \frac{k}{s^2 + k^2} \right\}$$

$$(e) \cos kt = \mathcal{L}^{-1} \left\{ \frac{s}{s^2 + k^2} \right\}$$

$$(f) \sinh kt = \mathcal{L}^{-1} \left\{ \frac{k}{s^2 - k^2} \right\}$$

$$(g) \cosh kt = \mathcal{L}^{-1} \left\{ \frac{s}{s^2 - k^2} \right\}$$

Example: Find the inverse Laplace transform.

$$\frac{1}{6} \mathcal{L}^{-1} \left\{ \frac{16}{s^4} \right\} \quad n=3$$

$$\frac{1}{3} \int 3e^{3x} dx$$
$$\frac{1}{3} e^{3x} + C$$

$$= \frac{1}{6} t^3$$

$$\mathcal{L}^{-1} \left\{ \frac{10s}{s^2+16} \right\} \text{ cosine}$$

$$= 10 \cos 4t$$

$$\mathcal{L}^{-1} \left\{ \frac{s+1}{s^2+2} \right\} = \mathcal{L}^{-1} \left\{ \frac{s}{s^2+2} \right\} + \frac{1}{\sqrt{2}} \mathcal{L}^{-1} \left\{ \frac{1 \cdot \sqrt{2}}{s^2+2} \right\}$$

$$= \cos \sqrt{2} t + \frac{1}{\sqrt{2}} \sin \sqrt{2} t$$

$$\int \frac{x+1}{x^2-4x} dx \quad \text{PFD}$$

$$s+1 = (A+B)s - 4A$$

$$\mathcal{L}^{-1} \left\{ \frac{s+1}{s(s-4)} \right\}$$

$$\frac{s+1}{s(s-4)} = \frac{A}{s} + \frac{B}{s-4} \rightarrow s+1 = A(s-4) + Bs$$

↑ constant ↑ linear

$$s=0: A = -\frac{1}{4}$$

$$s=4: B = \frac{5}{4}$$

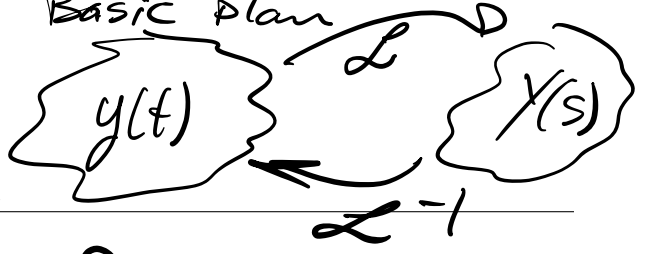
$$\mathcal{L}^{-1} \left\{ -\frac{1/4}{s} + \frac{5/4}{s-4} \right\}$$

$$= -\frac{1}{4} + \frac{5}{4} e^{4t}$$

Example: Use the Laplace transform to solve the Initial-Value Problem.

$$2 \frac{dy}{dt} + y = 0, \quad y(0) = -3$$

Basic Plan



Plan: Apply the Laplace transform

to both sides: $2 \mathcal{L} \left\{ \frac{dy}{dt} \right\} + \mathcal{L} \{ y \} = 0$

Transforms of derivatives

$$\mathcal{L} \{ f'(t) \} = \int_0^{\infty} e^{-st} f'(t) dt$$

$u = e^{-st} \quad dv = f'(t) dt$
 $du = -s e^{-st} dt \quad v = f(t)$

$$= \frac{f(t)}{e^{st}} \Big|_0^{\infty} + s \int_0^{\infty} e^{-st} f(t) dt$$

$$= -f(0) + s F(s)$$

$$\Rightarrow \mathcal{L} \{ f'(t) \} = s F(s) - f(0)$$

↑ from initial condition

Plan, cont: solve for $Y(s)$ and use the inverse Laplace transform to find y

$$2 \frac{dy}{dt} + y = 0, \quad y(0) = -3$$

$$2(sY(s) + 3) + Y(s) = 0$$

$$2sY(s) + Y(s) = -6 \Rightarrow Y(s) = -\frac{6}{2s+1}$$

$$Y(s) = -\frac{3}{s + 1/2} \Rightarrow y(t) = -3e^{-1/2 t}$$

$$\mathcal{L}\{f''(t)\} = \int_0^{\infty} e^{-st} f''(t) dt$$

$$u = e^{-st}$$

$$dv = f''(t) dt$$

$$= \frac{f'(t)}{e^{st}} \Big|_0^{\infty} + s \int_0^{\infty} e^{-st} f'(t) dt$$

$$= -f'(0) + s \mathcal{L}\{f'(t)\}$$

$$= -f'(0) + s [sF(s) - f(0)]$$

$$\mathcal{L}\{f''(t)\} = s^2 F(s) - s f(0) - f'(0)$$

Recall $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$

$$\text{Guess } \mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - f^{(n-1)}(0)$$

Theorem: Transform of a Derivative

If $f, f', \dots, f^{(n-1)}$ are continuous on $[0, \infty)$ and are of exponential order and if $f^{(n)}(t)$ is piecewise continuous on $[0, \infty)$, then

$$\mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - f^{(n-1)}(0),$$

where $F(s) = \mathcal{L}\{f(t)\}$.

Example: Use the Laplace transform to solve the Initial-Value Problem.

$$y'' - 4y' = 6e^{3t} - 3e^{-t}, \quad y(0) = 1, \quad y'(0) = -1$$

$$s^2 Y(s) - sy(0) - y'(0) - 4[sY(s) + 4y(0)] = \frac{6}{s-3} - \frac{3}{s+1}$$

$$(s^2 - 4s)Y(s) - s + 1 + 4 = \frac{6}{s-3} - \frac{3}{s+1}$$

$$s(s-4)Y(s) = \frac{6}{s-3} - \frac{3}{s+1} + s - 5$$

$$s(s-4)Y(s) = \frac{6(s+1) - 3(s-3) + (s-5)(s-3)(s+1)}{(s-3)(s+1)}$$

$$s(s-4)Y(s) = \frac{3s+15 + s^3 - 8s^2 + 15s + s^2 - 8s + 15}{(s-3)(s+1)}$$

$$Y(s) = \frac{s^3 - 7s^2 + 10s + 30}{s(s-4)(s-3)(s+1)}$$

$$\frac{s^3 - 7s^2 + 10s + 30}{s(s-4)(s-3)(s+1)} = \frac{A}{s} + \frac{B}{s-4} + \frac{C}{s-3} + \frac{D}{s+1}$$

$$s^3 - 7s^2 + 10s + 30 = A(s-4)(s-3)(s+1) + Bs(s-3)(s+1) + Cs(s-4)(s+1) + Ds(s-4)(s-3)$$

$$s=0 \quad 30 = 12A \Rightarrow A = 5/2$$

$$\underline{s=4} \quad 64 - 112 + 40 + 30 = 20B \Rightarrow B = \frac{11}{10}$$

We similarly find $C = -2$, $D = -\frac{3}{5}$

Wolfram alpha. com: $\text{Apart}(\text{(numerator)/denom})$

$$Y(s) = \frac{5/2}{s} + \frac{11/10}{s-4} - \frac{2}{s-3} - \frac{3/5}{s+1}$$

$$y(t) = \frac{5}{2} + \frac{11}{10} e^{4t} - 2e^{3t} - \frac{3}{5} e^{-t}$$